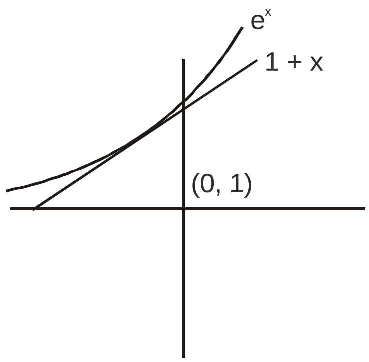
# EXERCISE – V

## HINTS & SOLUTIONS

**Sol.1** **(a)**




**(A)** $e^x < 1 + x$

**(B)** Let $f(x) = \ln(1 + x) - x$

$f'(x) = \dfrac{-x}{1+x}$

$f'(x) < 0$ for $x \in (0, 1)$

$f(x) < f(0)$

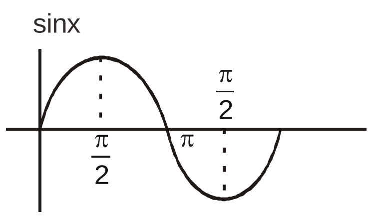
$\boxed{f(x) < 0}$

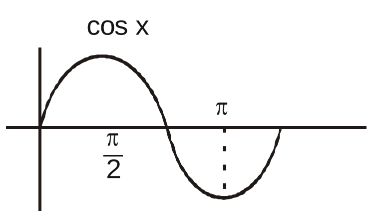
**(C)** $f(x) = \sin x - x$

$f'(x) = \cos x - 1 = -2\sin^2 x/2$

$f(x) \downarrow$ is $(0, 1)$

**(b)** S : is correct




Both are decreasing function.

R : $y = \sin x$

$y = \cos x$

$y' = \cos x$ decreasing

$y' = -\sin x \rightarrow$ increasing

R is incorrect.

**(c)** $f'(x) = e^x (x - 1)(x - 2)$

$\oplus \quad \Theta \quad \oplus$

f is $\downarrow$ is $(1, 2)$.

**Sol.2** **(a)** $f(x) = xe^{x(1-x)}$

$f'(x) = e^{x(1-x)} + xe^{x(1-x)}(1 - 2x)$

$= e^{x(1-x)}[1 + x - 2x^2]$

$= -e^{x(1-x)}(2x + 1)(x - 1) = 0$

$\Theta \quad \oplus \quad \oplus$

$-\dfrac{1}{2} \qquad 1$

$\downarrow$ is $\left(-\dfrac{1}{2}, 1\right)$

**(b)** $-1 \le p \le 1$

$4x^3 - 3x - p = 0$

assume $x = \cos\theta \Rightarrow \theta = \cos^{-1} x$

$\theta \in [0, \pi]$

$\cos 3\theta = p \Rightarrow 3\theta = \cos^{-1} p$

$0 \le 3\theta \le \pi$

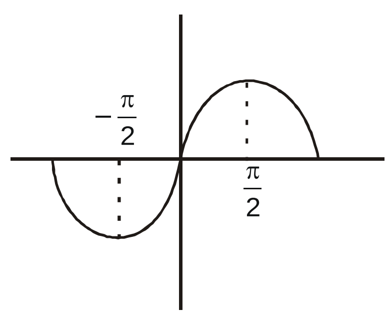
$0 \le \theta \le \dfrac{\pi}{3}$ $\qquad$ $3\theta = \cos^{-1} p$

$1 \ge \cos\theta \ge \dfrac{1}{2}$ $\qquad$ $\theta = \dfrac{1}{3}\cos^{-1} p$

$\cos^1 x = \dfrac{1}{3}\cos^1 P$

$\boxed{\dfrac{1}{2} \le x \le 1}$ $\qquad$ $x = \cos\left(\dfrac{1}{3}\cos^{-1} p\right)$

**Sol.3** $f(x) = 3\sin x - 4\sin^{-3} x$




$f(x) = \sin 3x$

$-\dfrac{\pi}{2} \le 3x \le \dfrac{\pi}{2}$

$-\dfrac{\pi}{6} \le x \le \dfrac{\pi}{6}$

Length of internal $= \dfrac{\pi}{6} - \left(-\dfrac{\pi}{6}\right) = \dfrac{\pi}{3}$

**Sol.4 (a)** $2(1-\cos x) < x^2, x \neq 0$ ...(1)

$\Rightarrow (1-\cos(\tan x)) < \dfrac{\tan^2 x}{2}$

P.T. $\sin(\tan x) \geq x\ \forall\ x \in \left[0, \dfrac{\pi}{4}\right]$

Let $f(x) = x - \sin(\tan x)$

$f'(x) = 1 - \cos(\tan x).\sec^2 x$

$= 1 - \cos(\tan x)(1 + \tan^2 x)$

$= 1 - \cos(\tan x) - \cos(\tan x)\tan^2 x < \dfrac{\tan^2 x}{2} -$

$\cos(\tan x)\tan^2 x$ by use equation (2)

$f'(x) < \tan^2 x \left(\dfrac{1}{2} - \cos(\tan x)\right)$

$< \tan^2 x \left(1 - \cos(\tan x) - \dfrac{1}{2}\right)$

$< \tan^2 x \left(\dfrac{\tan^2 x}{2} - \dfrac{1}{2}\right)$ by using equation (2)

$f'(x) < \dfrac{\tan^2 x}{2}(\tan^2 x - 1) <$ for $\forall\ x \in \left[0, \dfrac{\pi}{4}\right]$

$f(x) \downarrow \Rightarrow f(x) \leq f(0)$

$f(x) \leq 0$

$x - \sin(\tan x) \leq 0$

$\boxed{\sin(\tan x) \geq x}$

**Sol.4 (b) (i)** using mean values them, there exist $b \in (0,$ $4)$ such that $f'(b) = \dfrac{f(4) - f(0)}{4}$ .....(1)

Now $(f(4))^2 - (f(0))^2 = \left(\dfrac{f(4) - f(0)}{4}\right)(f(4) + f(0)) \times 4$

From equation (1)

$(f(4))^2 - (f(0))^2 = 4f'(b)(f(4) + f(0))$ .......(2)

Range of function f must contain the intarval $[f(0), f(4)]$ or $[f(4), f(0)]$

$\Rightarrow \left(\dfrac{f(0) + f(4)}{2}\right) \in$ Range of the function

$\Rightarrow f(a) = \dfrac{f(0) + f(4)}{2}$

Now from equation (2)

$(f(4))^2 - (f(0))^2 = 8 f(a) f'(b)$

Hence proved.

**(b) (ii)** Let $t = x^2 \Rightarrow dt = 2x\,dx$

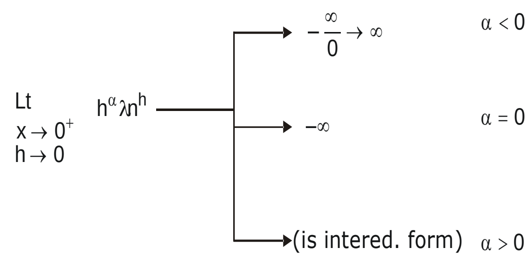
$$\int_0^4 f(t)\,dt = 2\int_0^2 f(x^2)\,dx = 2(2-0) f(c)$$

for some $c \in (0, 2)$ (using mean value them)

$$\int_0^4 f(t)\,dt = 2(f(c) + f(c))$$

$= 2(\alpha f(\alpha^2) + \beta f(\beta^2))$ where $\alpha = \beta = c$

**Sol.5** **(a)** $f(0^+) = f(0) = 0$




$\alpha$ should be greater than 0

**(b)** $\lim\limits_{x\to 0} \dfrac{f(x^2) - f(x)}{f(x) - f(0)} = \lim\limits_{x\to 0} \dfrac{f'(x^2)2x - f'(x)}{f'(x) - 0}$

$= \lim\limits_{x\to 0}\left[\dfrac{f'(x^2)2x}{f'(x)} - 1\right] = -1$

Aliter

function is increasing

so $f(4) > f(2)$ (for example)

If we calculate RHL & LHL than it should exist.

and $x \to 0^+$ $x^2 < x$

$f(x^2) < f(x) \Rightarrow f(x^2) - f(x) < 0$

will be negative.

**Sol.6** $p(x) = 51x^{101} - 2323x^{100} - 45x + 1035$

Let $f(x) = \int P(x)dx$

$f(x) = \dfrac{x^{102}}{2} - 23x^{101} - 45\dfrac{x^2}{2} + 1035x + c$

Now $f(45^{1/100}) = \dfrac{(45)x^2}{2} - (23)(45)x - 45\dfrac{x^2}{2} +$ $1035x + c$

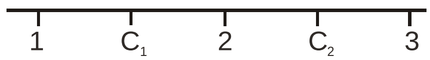
$f(45^{1/100}) = c$

$f(46) = \dfrac{(46)^{102}}{2} - 23(46)^{101} - 45\dfrac{(46)^2}{2} + (1035)46 = c$

By using rolle's tthem we can say that

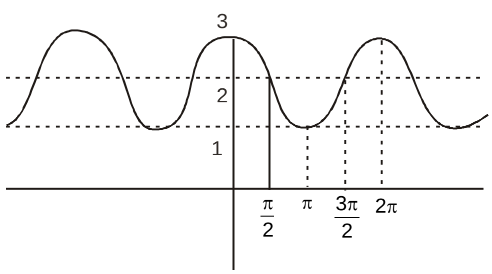
$f'(k) = 0$ for $k \in (45^{1/100}, 46)$
$51\,x^{101} - 23\,(k)^{100} - 45\,k + 1035 = 0$
$51\,x^{101} - 2323\,x^{100} - 45\,x + 1035 = 0$

**Sol.7** Let $g(x) = f(x) - x^2$
$g(1) = g(2) = g(3) = 0$
Rolle's them in [1, 2]
$g'(c_1) = 0$
$c_1 \in (1, 2)$
Rolle's them in [2, 3]
$g'(c_2) = 0;\ c_2 \in (2, 3)$



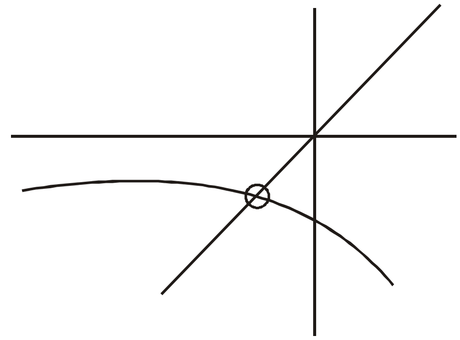

$g'(x) = f'(x) - 2x$
Apply Rolle's them is $[c_1c_2]$
$g''(c) = 0$ for some $c \in (c_1, c_2)$
$f''(c) - 2 = 0 \Rightarrow f''(c) = 2$ for some $c \in (1, 3)$
We can't assume $f(x) = x^2$
because
f(x) can be
$f(x) = x^2 + (x-1)(x-2)(x-3)\,\phi(x)$
$\phi(x)$ can be anthing.

**Sol.8 (a)** $f(x) = 2 + \cos x$ for $\forall\ x \in R$
Statement - I
$f'(c) = -\sin c$
$f(t) = 2 + \cos t \neq f(t + \pi)$




By graph there will exist at least onepoint where
$f'(c) = 0$
Statement - II
$f(x) = 2 + \cos x$
$f(t) = 2 + \cos t$
$f(t + 2\pi) = 2 + \cos t$
Statement-1 is correct and statement-II is also correct.
But statement-2 is not a correct explanation of statement-1.

**(b)** $f(x) = ke^x - x$
**(i)** $y = x$ and $y = ke^x$
If $k \leq 0$
one point



**(ii)** $ke^x - x = 0$
$y_1 = ke^x,\ y_2 = x$

$$\frac{dy_1}{dx} = ke^x \quad \frac{dy_2}{dx} = 1$$

$ke^x = 1$

$$e^x = \frac{1}{k} \Rightarrow x = \ell n\left(\frac{1}{k}\right)$$

$$y_1 = 1,\ y_2 = \ell n\left(\frac{1}{k}\right)$$

$$\ell n\left(\frac{1}{k}\right) = 1$$

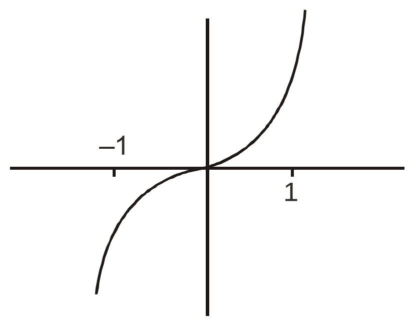
$$\ell nk = -1 \Rightarrow \boxed{k = \frac{1}{e}}$$

**(iii)** For two distinct root
$1 + \ell nk < 0\ (k > 0)$
$\ell nk < -1$

$$k < \frac{1}{e}$$

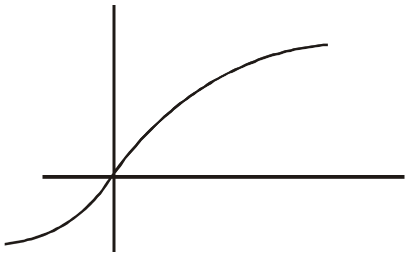
$$k \in \left(0, \frac{1}{e}\right)$$

**(c)** (A) $f(x) = x|x|$ $\begin{cases} x^2 & x \geq 0 \\ -x^2 & x < 0 \end{cases}$



**(B)** $f(x) = \sqrt{|x|} \rightarrow \sqrt{x} \;\; x \geq 0$

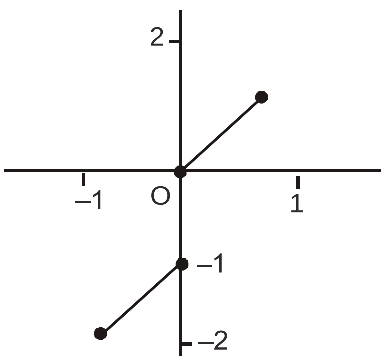
$y = \sqrt{x} \Rightarrow \boxed{x = y^2}$



**(C)** $f(x) = x + [x] \rightarrow x - 1 \quad -1 \leq x < 0$

$x \quad 0 \leq x < 1$

$2 \quad x = 1$



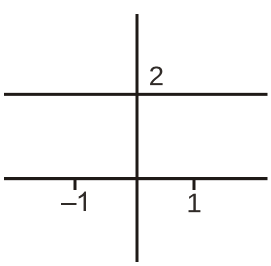

**(D)** $f(x) = |x - 1| + |x + 1|$

$x < -1$

$f(x) = |-x - x -| = -2x$

$-1 \leq x < 1 \qquad x \geq 1$

$f(x) = 1 - x + x + 1 = 2; \quad f(x) = x - 1 + x + 1 = 2x$




**Sol.9** **(a)** $g : (-\infty, \infty) \rightarrow \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$g(u) = 2\tan^{-1}(e^u) - \frac{\pi}{2}$

$g(-u) = 2\tan^{-1}\left(\frac{1}{e^u}\right) - \frac{\pi}{2}$

$= 2\cos^{-1} e^u - \frac{\pi}{2}$

$= 2\left(\frac{\pi}{2} - \tan^{-1} e^u\right) - \frac{\pi}{2}$

$g(-u) = \frac{\pi}{4} - 2\tan^{-1} e^u = -g(u)$

$g(u) \rightarrow$ odd function

Because $e^u$ is strictly $\uparrow$ function.

and $\tan^{-1}$ is also $\uparrow$ function

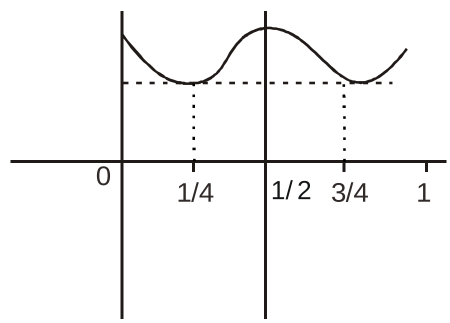
So g(u) will also $\uparrow$ function.

**(b)** $f'(x) \neq 0$ for $\forall\, x \in R$

$f(x) = f(1 - x) \qquad f'\left(\frac{1}{4}\right) = 0$

Replace $x \rightarrow \frac{1}{2} - x$

$f\left(\frac{1}{2} - x\right) = f\left(\frac{1}{2} + x\right)$




(A) By Rolle's theorem we can say that.

There are two points who

In between $\left[\frac{1}{-4}, \frac{1}{2}\right]$ and in between $\left[\frac{1}{2}, \frac{3}{4}\right]$

There will be atleast two points where $f''(x) = 0$

(B) $f'\left(\frac{1}{2}\right) = 0$ (By graph)

(C) $I = \int_{-1/2}^{1/2} f\left(x + \frac{1}{2}\right) \sin x \, dx$

$= -\int_{-1/2}^{1/2} f\left(\frac{1}{2} - x\right) \sin x \, dx \qquad$ (By)

$$I = -\int_{-1/2}^{1/2} f\left(\frac{1}{2}+x\right) \sin x\, dx$$

$I = -I \Rightarrow \boxed{I = 0}$

(D) $\int_0^{1/2} f(t)e^{\sin \pi t}dt = \int_{1/2}^{1} f(1-t)e^{\sin \pi t}dt$

Put 1 – t = z

$$= -\int_{1/2}^{0} f(z)e^{\sin \pi z}dz$$

$$= \int_0^{1/2} f(z)e^{\sin \pi z}dz$$

**Sol.10** $f(x) = x \cos \frac{1}{x}, x \geq 1$

$f'(x) = \cos \frac{1}{x} + \frac{1}{x} \cos \left(\frac{1}{x}\right) \to 1$ as $x \to \infty$

Also $f''(x) = \frac{1}{x^2} \sin \frac{1}{x} - \frac{1}{x^2} \sin \frac{1}{x} - \frac{1}{x^3} \cos \left(\frac{1}{x}\right)$

$= -\frac{1}{x^3} \cos \left(\frac{1}{x}\right) < 0$ for $x \geq 1$

f'(x) is decreasing is $[1, \infty)$

$f'(x + 2) < f'(x)$.

**Sol.11** $f'(x) = \frac{1}{x} + \sqrt{1+\sin x}$

f'(x) is not differentiable at

$\sin x = -1$ or $x = 2n\pi - \frac{\pi}{2}, n \in N$

$\ell n\, n \in (1, \infty)$ ; $f(x) > 0, f'(x) > 0$

consider $f(x) - f'(x)$

$= \ell n\, x + \int_0^x \sqrt{1+\sin t}\, dt - \frac{1}{x} - \sqrt{1+\sin x}$

$= \left(\int_0^x \sqrt{1+\sin t}\, dt - \sqrt{1+\sin x}\right) + \ell n\, x - \frac{1}{x}$

consider $g(x) = \int_0^x \sqrt{1+\sin t}\, dt - \sqrt{1+\sin x}$

It can be proved that $g(x) \geq 2\sqrt{2} - \sqrt{10}\ \forall x \in (0, \infty)$

Now there exists some $\alpha > 1$ such that

$\frac{1}{x} - \ell n\, x \leq 2\sqrt{2} - \sqrt{10}$ for all $x \in (\alpha, \infty)$ as

$\frac{1}{x} - \ell n\, x$ is strictly decreasing function

$$g(x) \geq \frac{1}{x} - \ell n\, x$$

Let $f(x) = (1 - x)^2 \sin^2 x + x^2$ for all $x \in IR$,

and let $g(x) = \int_1^x \left(\frac{2(t-1)}{t+1} - \ell n\, t\right) f(t)dt$ for all $x \in (1, \infty)$

**Sol.12 B**

$f(x) = (1-x)^2 \sin^2 x + x^2$

$g'(x) = \left(\frac{2(x-1)}{x+1} - \ell n\, x\right) f(x)$

$= \left(2 - \frac{4}{x+1} - \ell n\, x\right)$ (f(x) is positive)

Let $h = 2 - \ell n\, x - \frac{4}{x+1}$

$h'(x) = \frac{-1}{x} + \frac{4}{(x+1)^2} = \frac{-(x-1)^2}{x(x+1)^2}$

(always negative)

at x = 1 ; h = 0 so always negative

$\Rightarrow$ g'(x) always negative

**Sol.13 C**

$f(x) = 2(1 + x^2) - 2x = 2(1 + x^2 - x)$

$(1 - x)^2 \sin^2 x + x^2 = 2 + 2x^2 - 2x - (1 - x)^2$

$\cos^2 x = 1$, Not possible so P is false

for Q, $(1 - x)^2 \sin^2 x + x^2 = x + x^2 - 1/2$

$(1 - x)^2 \sin^2 x = x - 1/2$ obvious $x > 1/2$

$(1 - x)^2 \sin^2 x + 1/2 = x$

Let $y = (1 - x)^2 \sin^2 x - x + 1/2$

at $x = 1/2 \Rightarrow y = \frac{1}{4}\sin^2 \frac{1}{2}$ (positive)

at $x=1 \Rightarrow y = -\frac{1}{2}$ (negative) so Q is true